Diffusion in shear flows made easy: the Taylor limit

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G. I. Taylor (1953) gave a simple recipe for the calculation of contaminant dispersion in bounded shear flows at large times after discharge. He decomposed the concentration profile across the flow into a resolved (uniform) part, with an equilibrium (large-time) estimate for the unresolved part. Here an extended recipe is given to include greater resolution and earlier validity. At the two-equation level there is a close similarity to the slow-zone model posed by Chikwendu & Ojiakor (1985). Application is given to Poiseuille pipe flow and to a contraflowing parallel-plate heat exchanger.

1. Introduction

In a paper communicated to the Royal Society by G. I. Taylor, Townsend (1951) revealed the mechanism whereby velocity shear begins to pull out a heat or dye spot and leads to a rapidly increasing rate of dilution (see figure 1). These early stages of contaminant dispersion have led to much complicated mathematics. Taylor (1953) recognized that for bounded shear flows this shear dispersion mechanism continues to operate, even when the concentration has become nearly uniform across the flow. As long as there is some concentration variation across the flow, the different velocities in different parts of the flow provide an efficient mechanism for longitudinal dispersion.

The key to Taylor's analysis was the calculation of the residual concentration variation across the flow. He envisaged an eventual equilibrium between the tendency for shear to generate lateral concentration gradients by the rotation of longitudinal gradients, and the smoothing-out by lateral diffusion. The simplicity of Taylor's equilibrium analysis is illustrated in the next section, with a minor generalization to incorporate the effects of loss through the boundaries. The resulting expression for the effective longitudinal diffusivity, or shear dispersion coefficient, is not new; but, the calculation provides a framework for the more substantial generalization given in the remainder of the paper.

The time restrictions upon the applicability of the Taylor limit are quite stringent. At moderate times, the effective rate of longitudinal dilution is increasing towards the Taylor asymptote. Also, the longitudinal concentration profiles can develop marked and slowly decaying skewness (Chatwin 1970). The origins of these departures from the Taylor limit lie in the concentration variations across the flow: the rapidly changing longitudinal concentration gradient disturbs the cross-stream balance between shear and diffusion. Two-zone models allow directly for such departures from equilibrium through the use of two concentrations.

Mathematically, the pair of diffusion equations posed by Chikwendu & Ojiakor
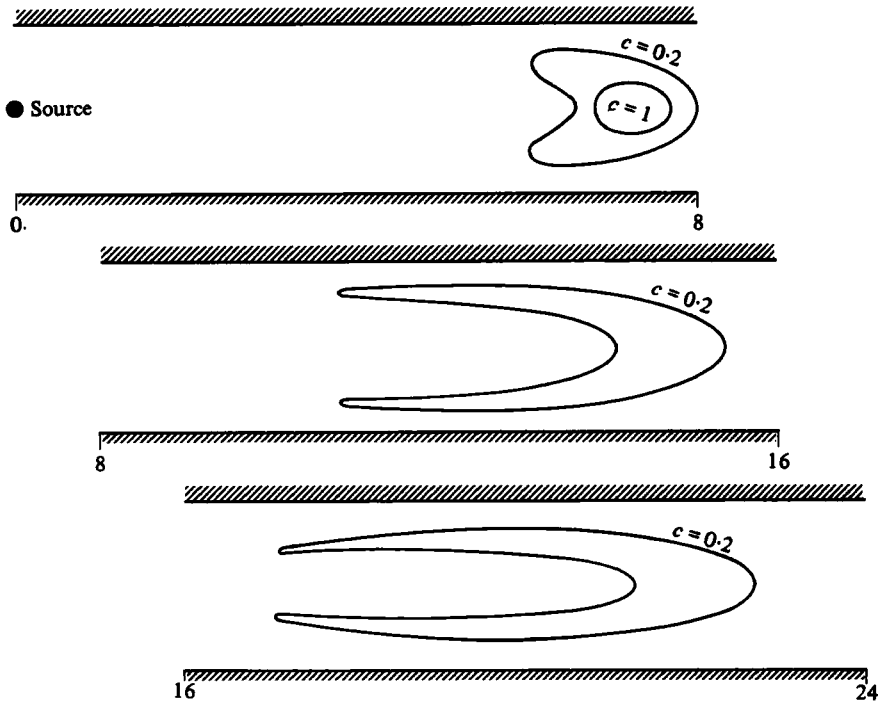
FIGURE 1. The severe distortion of a dye spot in the early stages of contaminant dispersion
(Smith 1981b, fig. 8).

(1985) seem to be a natural generalization of Taylor's single diffusion equation.
However, the arguments used by Chikwendu & Ojiakor to select where to split the
flow field into zones have no counterpart in Taylor's analysis. The purpose of the
present paper is to give a derivation based upon Taylor's ideas. In the process, minor
deficiencies of the Chikwendu & Ojiakor model are rectified (see Appendix A), and
a generalization is given to $N+1$ diffusion equations.

An important class of problems which are only accessible to analysis at the
two-equation level is transient behaviour in heat exchangers. A minimal description
requires two temperatures and velocities to describe the coupled system. Accordingly,
this paper includes a detailed application to parallel-plate heat exchangers.

2. Effect of boundary absorption upon longitudinal dispersion

For a high-Péclet-number flow, with absorption at the boundary, the advection–
diffusion equation takes the form

$$\partial_t c + u\,\partial_x c - \nabla \cdot (\kappa \nabla c) = q, \tag{2.1a}$$

with
$$\kappa \boldsymbol{n} \cdot \nabla c + \beta c = 0 \quad \text{on } \partial A. \tag{2.1b}$$

Here $c(x, y, z, t)$ is the concentration, $u(y, z)$ the longitudinal velocity, $\kappa(y, z)$ the
diffusivity, ∇ the transverse gradient operator $(0, \partial_y, \partial_z)$, $q(x, y, z, t)$ the source
strength, ∂A the boundary, $\boldsymbol{n}$ the outwards normal, and $\beta(y, z)$ a wall absorption
coefficient. The high-Péclet-number assumption (Taylor's condition A) permits us to
neglect a direct longitudinal diffusion term $\kappa\, \partial_x^2 c$, which is dominated by the effects

of longitudinal shear dispersion. Aris (1956) showed that, if required, this term can simply be added to the shear dispersion coefficient. The initial-value problem starting at $t = 0$ can be simulated by an appropriate delta-function source strength.

At large times after discharge the contaminant cloud will have become greatly elongated, and the $\partial_x c$-term in (2.12a) will be small. Thus, the concentration profile across the flow will equilibrate to the shape $\psi_0(y, z)$ of the lowest (single-signed) mode for the decay of concentration variations across the flow:

$$\nabla \cdot (\kappa \nabla \psi_0) + \lambda_0 \psi_0 = 0, \tag{2.2a}$$

with
$$\kappa \boldsymbol{n} \cdot \nabla \psi_0 + \beta \psi_0 = 0 \quad \text{on } \partial A, \tag{2.2b}$$

and
$$\overline{\psi_0^2} = 1. \tag{2.2c}$$

In the approach to this asymptotic profile, appropriate representations for the concentration and source strength are

$$c = c_0(x, t)\,\psi_0(y, z) + c' \quad \text{with } \overline{c'\psi_0} = 0, \tag{2.3a, b}$$

$$q = q_0(x, t)\,\psi_0(y, z) + q' \quad \text{with } q_0 = \overline{q\psi_0}. \tag{2.3c, d}$$

The basis of Taylor's (1953) work is to seek an evolution equation for the resolved part $c_0(x, t)$ of the concentration distribution, making equilibrium estimates of the unresolved residual c'. As shown in detail below, this leads to a gradient formula

$$c' = -f(y, z)\,\partial_x \bar{c}. \tag{2.4}$$

If we multiply (2.1a) by $\psi_0(x, y)$ and take the weighted average across the flow, then we obtain the evolution equation

$$\partial_t c_0 + u_{00}\,\partial_x c_0 + \lambda_0 c_0 + \overline{u\psi_0\,\partial_x c'} = q_0. \tag{2.5}$$

(The transverse diffusion term in (2.1a) gives rise to the $\lambda_0 c_0$ contribution.) Thus, the contaminant is carried along at the weighted average velocity $u_{00} = \overline{u\psi_0^2}$, and decays at the rate λ_0 appropriate to the lowest mode. Any longitudinal spreading is associated with the unresolved concentration variation c'. In the Taylor limit (2.4), this spreading is diffusive in character

$$\partial_t c_0 + u_{00}\,\partial_x c_0 + \lambda_0 c_0 - D\,\partial_x^2 c_0 = q_0. \tag{2.6a}$$

The effective longitudinal diffusivity D, or shear-dispersion coefficient, is given by the formula
$$D = \overline{u\psi_0 f}. \tag{2.6b}$$

The exact equation for c' is

$$\partial_t c' + u_{00}\,\partial_x c' - \nabla \cdot (\kappa \nabla c') = q' + (u_{00} - u)\,\psi_0\,\partial_x c_0 + \overline{u\psi_0\,\partial_x c'}\psi_0 + (u_{00} - u)\,\partial_x c', \tag{2.7a}$$

with
$$\kappa \boldsymbol{n} \cdot \nabla c' + \beta c' = 0 \quad \text{on } \partial A. \tag{2.7b}$$

The equilibrium estimate for c' is based upon the assumptions that q' has become negligible, $\partial_x c'$ is much less than $\partial_x c_0$, and that the rate of change of $\partial_x c_0$ is slow relative to the adjustment time for concentration variations across the flow (Taylor's condition B). Equivalently, enough time has elapsed that the concentration distribution is dominated by the lowest mode and is evolving slowly.

The advection and exponential decay of (lower case) c_0 interfere with this equilibrium requirement. Thus, as a temporary expedient we define (capitals)

$$c_0(x, y, z, t) = C_0(x - u_{00}t, y, z, t) \exp(-\lambda_0 t), \tag{2.8}$$

with corresponding definitions for C' and Q', so that the advection and exponential decay are explicitly accounted for. The temporary (capitals) version of $(2.7a)$ is

$$\partial_t C' - \lambda_0 C' - \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} C') = Q' + (u_{00} - u) \, \psi_0 \, \partial_x C_0 + \overline{u \psi_0 \, \partial_x C'} \psi_0 + (u_{00} - u) \, \partial_x C'. \quad (2.9)$$

The approximate solution for C', based upon the quasi-steadiness of $\partial_x C_0$ and the neglect of Q', $\partial_x C'$, is

$$C' = -f(y, z) \, \partial_x C_0 \quad\quad\quad\quad (2.10a)$$

$$c' = -f \partial_x c_0 \quad\quad\quad\quad (2.10b)$$

where the auxiliary function f describes the balance between shear and diffusion

$$\boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} f) + \lambda_0 f = (u_{00} - u) \, \psi_0, \quad\quad (2.11a)$$

with

$$\kappa \boldsymbol{n} \cdot \boldsymbol{\nabla} f + \beta f = 0 \quad \text{on } \partial A, \quad\quad (2.11b)$$

and

$$\overline{f \psi_0} = 0. \quad\quad\quad\quad (2.11c)$$

The normalization $(2.11c)$ restates that c' does not contribute to the lowest modal component of the concentration (equation (2.5)).

For flows with no z-dependence, $(2.11a)$ is an ordinary differential equation and can be integrated explicitly (Smith 1986, appendix B). The resulting formula for the shear-dispersion coefficient is

$$D = \frac{1}{h} \int_0^h \frac{1}{\kappa \psi_0^2} \left[\int_0^y (u - u_{00}) \, \psi_0^2 \, \mathrm{d}y' \right]^2 \mathrm{d}y, \quad\quad (2.12)$$

where h is the width (or depth) in the y-direction. The concentration profile ψ_0 modifies the weighting given to different parts of the flow. However, the features emphasized by Taylor (1953) remain apparent, i.e. inverse dependence upon κ, and quadratic dependence upon the velocity shear.

An alternative expression for D can be derived if we introduce the higher transverse diffusion modes:

$$\boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \psi_m) + \lambda_0 \psi_m = 0, \quad\quad (2.13a)$$

$$\kappa \boldsymbol{n} \cdot \boldsymbol{\nabla} \psi_m + \beta \psi_m = 0 \quad \text{on } \partial A, \quad\quad (2.13b)$$

$$0 \leqslant \lambda_0 < \lambda_1 < \lambda_2 < \ldots, \quad\quad (2.13c)$$

$$\overline{\psi_m^2} = 1, \quad \overline{\psi_m \psi_n} = 0 \quad \text{for } m \neq n. \quad\quad (2.13d)$$

Also, we define the velocity coefficients

$$u_{mn} = \overline{u \psi_m \psi_n}. \quad\quad\quad\quad (2.13e)$$

The solution for f can be written

$$f = \sum_{n=1}^{\infty} \frac{u_{0n}}{\lambda_n - \lambda_0} \psi_n(y, z), \quad\quad (2.14)$$

and the resulting summation for D is

$$D = \sum_{n=1}^{\infty} \frac{u_{0n}^2}{\lambda_n - \lambda_0}. \quad\quad\quad\quad (2.15)$$

As remarked in the Introduction, the above results are not new. A variety of derivations have been given by Sankarasubramanian & Gill (1973), Lungu & Moffatt (1982), Smith (1983), Barton (1984). The present derivation is faithful to the work of Taylor (1953), and prepares the way for the subsequent generalization. A forceful argument for the need for an improved model has been given by Fischer *et al.* (1979,

§5.5): when there is loss of contaminant there may be a negligible amount of contaminant left in the flow by the time that the Taylor limit is applicable!

3. Two-mode approximation

The Taylor limit (2.10a, b) can be inadequate if either $\partial_x C'$ is not small relative to $\partial_x C_0$, or $\partial_x C_0$ changes too rapidly. Both these facets can be improved if we extend the representation (2.3) to encompass another mode:

$$c \approx c_0(x,t)\,\psi_0(y,z) + c_1(x,t)\,\psi_1(y,z) + c_0' + c_1', \tag{3.1a}$$

with

$$\overline{c_j'\psi_0} = \overline{c_j'\psi_1} = 0; \tag{3.1b}$$

$$q = q_0(x,t)\,\psi_0(y,z) + q_1(x,t)\,\psi_1(y,z) + q', \tag{3.1c}$$

with

$$q_0 = \overline{q\psi_0}, \quad q_1 = \overline{q\psi_1}. \tag{3.1d}$$

The linearity of (2.1a) makes it natural to split c' into contributions associated with c_0 and c_1. The reasons why we expect improvement are that the unresolved part of the concentration has been made smaller, and the relevant timescale for equilibrium has been shifted from the first mode $1/\lambda_1$ to the second mode $1/\lambda_2$. In view of the previous calculation, we can anticipate that the equilibrium estimates for c_j' again have a gradient form

$$c_0' = -f_0(y,z)\,\partial_x c_0, \quad c_1' = -f_1(y,z)\,\partial_x c_1. \tag{3.2a, b}$$

The ψ_0 and ψ_1 weighted averages of (2.1a) yield the coupled pair of equations

$$\partial_t c_0 + u_{00}\,\partial_x c_0 + \lambda_0 c_0 = q_0 - u_{01}\,\partial_x c_1 - \overline{u\psi_0\,\partial_x c_0'} - \overline{u\psi_0\,\partial_x c_1'}, \tag{3.3a}$$

$$\partial_t c_1 + u_{11}\,\partial_x c_1 + \lambda_1 c_{10} = q_1 - u_{01}\,\partial_x c_0 - \overline{u\psi_1\,\partial_x c_0'} - \overline{u\psi_1\,\partial_x c_1'}. \tag{3.3b}$$

The modes have their own advection velocities u_{jj} and decay rates λ_j, with coupling via the cross-terms $u_{jk}\,\partial_x c_k$. The non-equilibrium Telegraph-equation model proposed by Smith (1981a) corresponds to neglecting the c_j' terms.

In the extended Taylor limit (3.2a, b), we have a pair of diffusion equations:

$$\partial_t c_0 + u_{00}\,\partial_x c_0 + u_{01}\,\partial_x c_1 + \lambda_0 c_0 = D_{00}\,\partial_x^2 c_0 + D_{01}\,\partial_x^2 c_1 + q_0, \tag{3.4a}$$

$$\partial_t c_1 + u_{01}\,\partial_x c_0 + u_{11}\,\partial_x c_1 + \lambda_1 c_1 = D_{10}\,\partial_x^2 c_0 + D_{11}\,\partial_x^2 c_1 + q_1, \tag{3.4b}$$

where

$$D_{00} = \overline{u\psi_0 f_0}, \quad D_{01} = \overline{u\psi_0 f_1}, \quad D_{10} = \overline{u\psi_1 f_0}, \quad D_{11} = \overline{u\psi_1 f_1}. \tag{3.4c-f}$$

The exact equation for c_j' is

$$\partial_t c_j' + u_{jj}\,\partial_x c_j' - \nabla\cdot(\kappa\nabla c_j') = [u_{j0}\,\psi_0 + u_{j1}\,\psi_1 - u\psi_j]\,\partial_x c_j + \overline{u\psi_0\,\partial_x c_j'}\,\psi_0$$

$$+ \overline{u\psi_1\,\partial_x c_j'}\,\psi_1 + (u_{jj} - u)\,\partial_x c_j' + q_j', \tag{3.5a}$$

with

$$\kappa\boldsymbol{n}\cdot\nabla c_j' + \beta c_j' = 0 \quad \text{on } \partial A. \tag{3.5b}$$

The occurrence of u_{jj} on the left-hand side of (3.5a) anticipates the fact that u_{jj} is the natural velocity associated with the ψ_j mode.

The removal of advection and decay proceeds as in the previous section. Thus, we again use capital-letter quantities C_j, C_j', Q_j:

$$c_j' = C_j'(x - u_{jj}t, y, z, t)\exp(-\lambda_j t). \tag{3.6}$$

The temporary (capitals) version of (3.5a) is

$$\partial_t C_j' - \lambda_j C_j' - \nabla\cdot(\kappa\nabla C_j') = [u_{j0}\,\psi_0 + u_{j1}\,\psi_1 - u\psi_j]\,\partial_x C_j + \overline{u\psi_0\,\partial_x C_j'}\,\psi_0$$

$$+ \overline{u\psi_1\,\partial_x C_j'}\,\psi_1 + (u_{jj} - u)\,\partial_x C_j' + Q_j'. \tag{3.7}$$

We now make the equilibrium assumption: that $\partial_x C_j$ can be treated as being steady, and that Q'_j, $\partial_x C'_j$ can be neglected (i.e. enough time, say $6/\lambda_2$, has elapsed that the concentration distribution is dominated by the ψ_0 and ψ_1 modes). Hence we make the approximations

$$C'_j = -f_j \partial_x C_j, \tag{3.8a}$$

i.e.

$$c'_j = -f_j \partial_x c_j \tag{3.8b}$$

where the auxiliary functions $f_j(y, z)$ satisfy the equations

$$\nabla \cdot (\kappa \nabla f_j) + \lambda_j f_j = u_{j0} \psi_0 + u_{j1} \psi_1 - u \psi_j, \tag{3.9a}$$

with

$$\kappa n \cdot \nabla f_j + \beta f_j = 0 \quad \text{on } \partial A, \tag{3.9b}$$

and

$$\overline{f_j \psi_j} = 0. \tag{3.9c}$$

The normalization (3.9d) only ensures that C'_0 does not contribute to the ψ_0 component of the concentration (with a complementary property for C'_1 with respect to ψ_1). For compatibility with (3.1b) we need the stronger result that C'_0 has no ψ_1 component. This property of equations (3.9a–d) confirms the appropriateness of the split representation (3.6).

In terms of the eigenmodes ψ_m the functions f_0, f_1 can be written

$$f_0 = \sum_{n=2}^{\infty} \frac{u_{0n}}{\lambda_n - \lambda_0} \psi_n(y, z), \quad f_1 = \sum_{n=2}^{\infty} \frac{u_{1n}}{\lambda_n - \lambda_1} \psi_n(y, z). \tag{3.10a, b}$$

This enables us to demonstrate the positivity of the diagonal shear-dispersion coefficients D_{00}, D_{11} and the inequality of the off-diagonal terms D_{01}, D_{10}:

$$D_{00} = \sum_{n=2}^{\infty} \frac{u_{0n}^2}{\lambda_n - \lambda_0}, \quad D_{01} = \sum_{n=2}^{\infty} \frac{u_{0n} u_{1n}}{\lambda_n - \lambda_1}, \tag{3.11a, b}$$

$$D_{10} = \sum_{n=2}^{\infty} \frac{u_{1n} u_{0n}}{\lambda_n - \lambda_0}, \quad D_{11} = \sum_{n=2}^{\infty} \frac{u_{1n}^2}{\lambda_n - \lambda_1}. \tag{3.11c, d}$$

We remark that for the initial-value problem, the q_0, q_1 forcing terms in equations (3.4a, b) imply that the correct initial data are the $c_0(x, 0)$, $c_1(x, 0)$ profiles.

4. $N+1$ diffusion equations

The extension to an $(N+1)$-mode approximation is now obvious. The representation for the concentration is

$$c = \sum_{j=0}^{N} (c_j(x, t) \psi_j(y, z) - f_j(y, z) \partial_x c_j). \tag{4.1}$$

The auxiliary functions $f_j(y, z)$ satisfy the equations

$$\nabla \cdot (\kappa \nabla f_j) + \lambda_j f_j = \sum_{k=0}^{N} u_{jk} \psi_k - u \psi_j, \tag{4.2a}$$

with

$$\kappa n \cdot \nabla f_j + \beta f_j = 0 \quad \text{on } \partial A, \tag{4.2b}$$

and

$$\overline{f_j \psi_j} = 0. \tag{4.2c}$$

The coupled diffusion equations are

$$\partial_t c_j + \sum_{k=0}^{N} u_{jk} \partial_x c_k + \lambda_j c_j = \sum_{k=0}^{N} D_{jk} \partial_x^2 c_k + q_j, \tag{4.3a}$$

where

$$D_{jk} = \overline{u\psi_j f_k} = \sum_{n=N+1}^{\infty} \frac{u_{jn} u_{kn}}{\lambda_n - \lambda_k}. \tag{4.3b}$$

As the number of modes is increased, the smaller the value of the dispersion coefficients D_{jk} becomes. This is because the unresolved concentration is less and less important.

The self-consistency of the different levels of truncation can be seen if we cease to resolve the ψ_N mode. We split the equation for c_N into component parts c_k' associated with forcing by the lower c_k modes:

$$\partial_t c_k' + u_{kk} \partial_x c_k' + \lambda_N c_k'$$
$$= -u_{NK} \partial_x c_k + (u_{kk} - u_{NN}) \partial_x c_k' + D_{Nk} \partial_x^2 c_k + D_{NN} \partial_x^2 c_k' + q_k'. \tag{4.4}$$

In the now familiar way, we change to the capital C variables in which decay and advection have been accounted for:

$$\partial_t C_k' + (\lambda_N - \lambda_k) C_k'$$
$$= -u_{NK} \partial_x C_k + (u_{kk} - u_{NN}) \partial_x C_k' + D_{Nk} \partial_x^2 C_k + D_{NN} \partial_x^2 C_k' + Q_k'. \tag{4.5}$$

On the premise that $\partial_x C_k'$, $\partial_x^2 C_k'$ are small relative to $\partial_x C_k$, $\partial_x^2 C_k$ and that the time variations are slow relative to $1/(\lambda_N - \lambda_k)$, we make the equilibrium approximation

$$C_k' = -\frac{u_{NK}}{\lambda_N - \lambda_k} \partial_x C_k + \frac{D_{Nk}}{\lambda_N - \lambda_k} \partial_x^2 C_k, \tag{4.6a}$$

i.e.

$$C_N = -\sum_{k=0}^{N-1} \frac{u_{NK}}{\lambda_N - \lambda_k} \partial_x C_k + \sum_{k=0}^{N-1} \frac{D_{Nk}}{\lambda_N - \lambda_k} \partial_x^2 C_k. \tag{4.6b}$$

If we make the stronger assumption that $\partial_x C_k$ could be regarded as constant with respect to both x and t, then the $\partial_x^2 C_k$ terms would be absent in (4.6b).

Substituting for c_N into (4.3a) we have

$$\partial_t c_j + \sum_{k=0}^{N-1} u_{jk} \partial_x c_k + \lambda_j c_j - \sum_{k=0}^{N-1} \left[D_{jk} + \frac{u_{Nk} u_{Nj}}{\lambda_N - \lambda_k} \right] \partial_x^2 c_k$$
$$= -\sum_{k=0}^{N-1} \left\{ \frac{u_{Nk} D_{jN} + u_{jN} D_{Nk}}{\lambda_N - \lambda_k} \right\} \partial_x^3 c_k + q_k. \tag{4.7}$$

To recover the N-mode version of (4.3a, b) it suffices that we neglect the $\partial_x^3 c_k$ terms. As time goes by the contaminant cloud gets elongated, and the higher (third) x-derivatives decay faster than the lower (second) derivatives. Thus, the neglect of the ψ_N mode is justified if sufficient time (of order $1/\lambda_N$) has elapsed. The more modes, the earlier the time that the equations are applicable.

For a given application the appropriate number of modes needed depends upon how soon after discharge the results are required. For t in the range

$$6/\lambda_{N+1} < t < 6/\lambda_N, \tag{4.8}$$

it would be appropriate to explicitly represent the modes $\psi_0, \ldots, \psi_N$, i.e. to use $N+1$ diffusion equations. With more than six e-folding times, the higher modes $\psi_{N+1}, \ldots$ can be regarded as being in equilibrium.
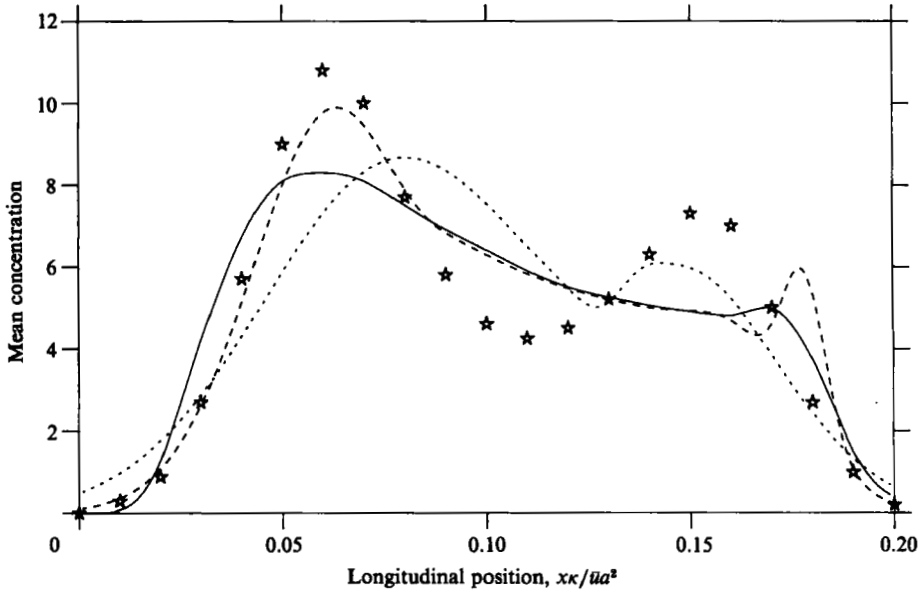
FIGURE 2. Numerical test of the three-equation (—–—) as compared with the two-equation (......) solution and the full numerical solution (——) for Poiseuille pipe flow. The stars show Chikwendu's (1986) results for the slow-zone model.

5. Poiseuille pipe flow

As a numerical illustration of the coupled-diffusion-equation model, it is appropriate that we should follow Taylor (1953) and investigate Poiseuille pipe flow with impermeable boundaries. The velocity profile, eigenmodes and velocity coefficients are

$$u = 2\overline{u}\left[1 - \frac{r}{a^2}\right], \quad \psi_m = \frac{J_0(\gamma_m(r/a))}{J_0(\gamma_m)}, \tag{5.1a, b}$$

$$J_0'(\gamma_m) = 0, \quad \lambda_m = \frac{\gamma_m^2 \kappa}{a^2}, \tag{5.1c, d}$$

$$u_{00} = \overline{u}, \quad u_{mm} = \tfrac{4}{3}\overline{u}, \tag{5.1e, f}$$

$$u_{0m} = -\frac{8}{\gamma_m^2}\overline{u}, \quad u_{mn} = -\frac{8(\gamma_m^2 + \gamma_n^2)\,\overline{u}}{(\gamma_m^2 - \gamma_n^2)^2}. \tag{5.1g, h}$$

Performing the summations (3.11a–d) we find that the numerical values of the coefficients in (3.4a, b) are

$$u_{00} = \overline{u}, \quad u_{01} = -0.5449\overline{u}, \quad u_{11} = \tfrac{4}{3}\overline{u} \tag{5.2a, b, c}$$

$$\lambda_0 = 0, \quad \lambda_1 = \frac{14.68k}{a^2} \tag{5.2d, e}$$

$$D_{00} = 0.0006112\frac{\overline{u}^2 a^2}{\kappa}, \quad D_{01} = 0.002144\frac{\overline{u}^2 a^2}{\kappa}, \tag{5.2f, g}$$

$$D_{10} = 0.001526\frac{\overline{u}^2 a^2}{\kappa}, \quad D_{11} = 0.005508\frac{\overline{u}^2 a^2}{\kappa}. \tag{5.2h, i}$$
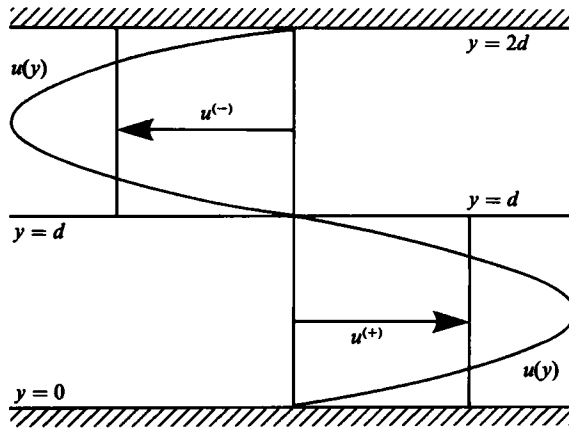
FIGURE 3. The velocity profile for a contraflowing parallel-plate heat exchanger.

Figure 2 compares the solution at $t = 0.1a^2/\kappa$ for a uniform discharge of length $0.012a^2\overline{u}/\kappa$ with the numerical solution computed by Gill & Ananthakrishnan (1967). Longitudinal diffusion was allowed for by the addition of $4 \times 10^{-6}\,\overline{u}^2a^2/\kappa$ to the diagonal diffusion coefficients D_{jj}. The NAG routine D03 PGF was used to solve the coupled diffusion equations. At this early time after discharge the profile for c_0 is markedly non-Gaussian. However, the coupled-diffusion-equation model does manage to reproduce the qualitative features, and is more accurate than single-equation models (Smith 1981a, figure 4). The criterion (4.8) would suggest that three diffusion equations would be more appropriate, and accordingly figure 2 includes results for the three-equation model.

After this paper had been submitted, I learned that Chikwendu (1986) had applied the slow-zone method to Poiseuille pipe flow. Numerical results from his figure 8 are shown as the stars in the present figure 2. The shape of the profile is qualitatively similar to, but less accurate than, the present two-equation model.

6. Parallel-plate heat exchanger

As an example to which a two-zone model seems particularly natural, we consider transient heat exchange between contraflowing plane Poiseuille flows (see figure 3):

$$u = 6\overline{u}\left[\left(\frac{y}{d}\right) - \left(\frac{y}{d}\right)^2\right] \quad \text{for } 0 \leqslant y \leqslant d, \tag{6.1a}$$

$$u = -6\overline{u}\left[3\left(\frac{y}{d}\right) - \left(\frac{y}{d}\right)^2 - 2\right] \quad \text{for } d \leqslant y \leqslant 2d, \tag{6.1b}$$

We take the outer boundaries to be perfectly insulated:

$$\kappa\,\partial_y c = 0 \quad \text{on } y = 0.2d, \tag{6.2a}$$

and across the thin boundary between the flows we assume there is no thermal resistance:

$$c \quad \text{and} \quad \kappa\,\partial_y c \quad \text{continuous across } y = d. \tag{6.2b}$$

Nunge & Gill (1965) have investigated the steady state, including the additional complication of differing bulk velocities in the two flows.
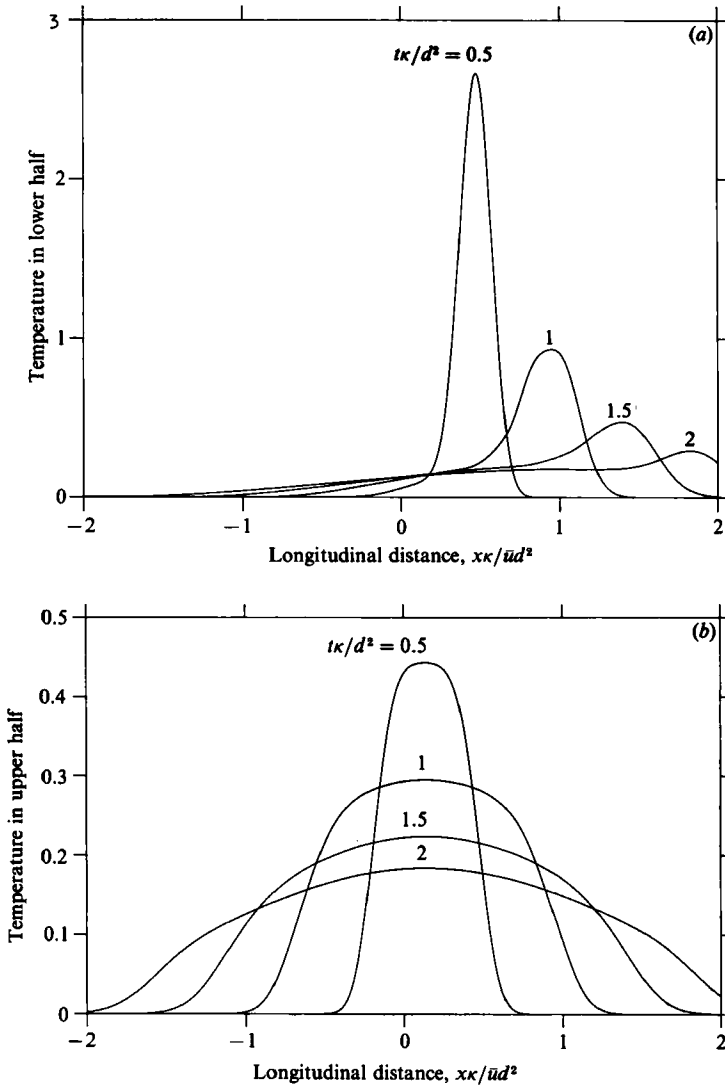
FIGURE 4. The longitudinal temperature distributions in the two halves of a contraflowing heat exchanger for an initial hot spot in just the forward-moving zone: (a) forward-moving zone; (b) backward-moving zone.

The lowest mode for the composite system is

$$\psi_0 = 1, \quad \lambda_0 = 0, \qquad\qquad (6.3\,a, b)$$

and corresponds to equalization of temperature between the two flows. For constant κ (laminar flows) the next mode is

$$\psi_1 = \sqrt{2} \cos\left(\tfrac{1}{2}\pi \frac{y}{d}\right) \quad \text{for } 0 < y < 2d, \qquad\qquad (6.4\,a)$$

$$\lambda_1 = \tfrac{1}{4}\pi^2 \frac{\kappa}{d^2}, \qquad\qquad (6.4\,b)$$

and describes the equilibration of the temperature between the flows.

The symmetry/anti-symmetry of the velocity profile and the modes implies that the diagonal velocity coefficients u_{00}, u_{11} are identically zero. Thus, from (5.2a, b) we infer that the effective velocities for the two zones are $\pm u_{01}$, where

$$u_{01} = \frac{\overline{u}24\sqrt{2}[4-\pi]}{\pi^3} = 0.9396\overline{u}. \tag{6.5}$$

Despite the absence of thermal resistance between the two flows, the effective velocities are remarkably close to the respective bulk velocities $\pm\overline{u}$.

Symmetry considerations also permit us to deduce that the off-diagonal dispersion coefficients D_{01}, D_{10} are zero. To calculate the diagonal terms, we first calculate the shape functions f_0, f_1 for the unresolved part of the concentration profile:

$$\frac{\kappa}{\overline{u}d^2}f_0 = \frac{1}{2} - \left[\frac{y}{d}\right]^3 + \frac{1}{2}\left[\frac{y}{d}\right]^4 - \frac{192}{\pi^5}[4-\pi]\cos\left[\tfrac{1}{2}\pi\frac{y}{d}\right], \tag{6.6a}$$

$$\frac{\kappa}{\overline{u}d^2}f_1 = \frac{96\sqrt{2}}{\pi^5}[4-\pi] - \left\{\frac{24}{\pi^4} + \frac{6}{\pi^2}\left(\frac{y}{d}\right) - \frac{6}{\pi^2}\left(\frac{y}{d}\right)^2\right\}\sqrt{2}\cos\left[\tfrac{1}{2}\pi\frac{y}{d}\right]$$

$$+ \left\{\frac{12}{\pi^3} - \frac{12}{\pi^3}\left(\frac{y}{d}\right) - \frac{3}{\pi}\left(\frac{y}{d}\right)^2 + \frac{2}{\pi}\left(\frac{y}{d}\right)^3\right\}\sqrt{2}\sin\left[\tfrac{1}{2}\pi\frac{y}{d}\right]. \tag{6.6b}$$

Across $y = d$ the extension of f_0 is anti-symmetric and f_1 is symmetric. Evaluating the integrals (3.4c–f) we arrive at the results

$$D_{00} = \frac{\overline{u}^2d^2}{\kappa}\left\{\frac{13}{35} - \frac{4608}{\pi^8}[4-\pi]^2\right\} = 0.01358\frac{\overline{u}^2d^2}{\kappa}, \tag{6.7a}$$

$$D_{11} = \frac{\overline{u}^2d^2}{\kappa}\left\{\frac{4608}{\pi^8}[4-\pi]^2 - \frac{6}{5\pi^2} - \frac{36}{\pi^4} + \frac{144}{\pi^6}\right\} = 0.01647\frac{\overline{u}^2d^2}{x}. \tag{6.7b}$$

Figure 4(a) shows the advance and dispersion of a unit temperature pulse in the forward-moving zone. Figure 4(b) shows the corresponding response in the initially unheated backward-moving zone. The solutions were obtained using the NAG routine D03 PGF.

In view of the relationship (2.15), (3.11a) between the one- and two-diffusion-equation dispersion coefficients, we can infer from (6.5) and (6.7a) that

$$D = \frac{13\overline{u}^2d^2}{35\kappa} = 0.3714\frac{\overline{u}^2d^2}{\kappa}. \tag{6.8}$$

Thus, the 'Taylor' shear-dispersion coefficient for the entire contraflowing system is an order of magnitude larger than the coefficients for the subsystems. This can be seen in figure 4(a), where the rate of spreading increases markedly once there has been significant exchange between the two zones.

7. Concluding remarks

The title of this paper makes the contention that in the Taylor limit the investigation of contaminant dispersion becomes easy. This claim rests on three points. First, that the decomposition of the concentration field into resolved and equilibrium parts makes the timescale limitations quite explicit. Second, the integrals (3.4c–f) or the series (3.11a–d) for the shear-dispersion coefficients are straight-forward to evaluate. Finally, constant-coefficient diffusion equations are much more familiar than some of the equations that have been advocated for the modelling of

contaminant dispersion. Indeed, as befits the Taylor centenary year in which this paper was completed, the extension is in the spirit of G. I. Taylor's (1953) research.

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Appendix A. Modes and zones

Although we have succeeded in deriving a pair of diffusion equations (3.4a, b) to describe shear dispersion, they do not have the form of the two-zone equations posed by Chikwendu & Ojiakor (1985). An obvious source of difference is that modes are associated with a decay rate λ_j, whereas zones are associated with a velocity $u^{(+)}, u^{(-)}$.

The eigenvelocities of the u_{ij} symmetric matrix are

$$u^{(+)}, u^{(-)} = \tfrac{1}{2}(u_{00} + u_{11}) \pm [u_{01}^2 + \tfrac{1}{4}(u_{11} - u_{00})^2]^{\frac{1}{2}}. \tag{A 1}$$

The relative thickness of the two zones are $\tfrac{1}{2}(1 - \xi), \tfrac{1}{2}(1 + \xi)$ where the asymmetry coefficient ξ is defined by

$$\xi = \frac{\tfrac{1}{2}(u_{11} - u_{00})}{[u_{01}^2 + \tfrac{1}{4}(u_{11} + u_{00})^2]^{\frac{1}{2}}}, \tag{A 2a}$$

i.e.

$$u^{(+)}, u^{(-)} = u_{00} + (\xi \pm 1)[u_{01}^2 + \tfrac{1}{4}(u_{11} - u_{00})^2]^{\frac{1}{2}}. \tag{A 2b}$$

In terms of the model concentrations c_0, c_1 we define the zone concentrations

$$c^{(+)} = c_0 + c_1 \operatorname{sgn}(u_{01}) \left[\frac{1 + \xi}{1 - \xi} \right]^{\frac{1}{2}}, \tag{A 3a}$$

$$c^{(-)} = c_0 - c_1 \operatorname{sgn}(u_{01}) \left[\frac{1 - \xi}{1 + \xi} \right]^{\frac{1}{2}}. \tag{A 3b}$$

The appropriate linear combinations of equations (3.4a, b) yield the two-zone equations

$$\partial_t c^{(+)} + u^{(+)} \partial_x c^{(+)} + \lambda_0 c^{(+)} + (\lambda_1 - \lambda_0) \tfrac{1}{2}(1 + \xi)(c^{(+)} - c^{(-)}) = D^{++} \partial_x^2 c^{(+)} + D^{+-} \partial_x^2 c^{(-)}, \tag{A 4a}$$

$$\partial_t c^{(-)} + u^{(-)} \partial_x c^{(-)} + \lambda_0 c^{(-)} - (\lambda_1 - \lambda_0) \tfrac{1}{2}(1 - \xi)(c^{(+)} - c^{(-)}) = D^{-+} \partial_x^2 c^{(+)} - D^{--} \partial_x^2 c^{(-)}. \tag{A 4b}$$

The new diffusion coefficients are given by the somewhat awkward formulae

$$D^{++} = \tfrac{1}{2}(1 - \xi) D_{00} + \tfrac{1}{2}(1 + \xi) D_{11} + \tfrac{1}{2}[1 - \xi^2]^{\frac{1}{2}} \operatorname{sgn}(u_{01})(D_{01} + D_{10}), \tag{A 5a}$$

$$D^{--} = \tfrac{1}{2}(1 + \xi) D_{00} + \tfrac{1}{2}(1 - \xi) D_{11} + \tfrac{1}{2}[1 - \xi^2]^{\frac{1}{2}} \operatorname{sgn}(u_{01})(D_{01} + D_{10}), \tag{A 5b}$$

$$D^{+-} = \tfrac{1}{2}(1 + \xi)(D_{00} - D_{11}) + \frac{1}{2}\left[\frac{1 + \xi}{1 - \xi} \right]^{\frac{1}{2}} \operatorname{sgn}(u_{01})\{(1 + \xi) D_{10} - (1 - \xi) D_{01}\}, \tag{A 5c}$$

$$D^{-+} = \tfrac{1}{2}(1 - \xi)(D_{00} - D_{11}) + \frac{1}{2}\left[\frac{1 - \xi}{1 + \xi} \right]^{\frac{1}{2}} \operatorname{sgn}(u_{01})\{(1 + \xi) D_{01} - (1 - \xi) D_{10}\}. \tag{A 5d}$$

In their model, Chikwendu & Ojiakor (1985) overlooked the possibility of off-diagonal dispersion coefficients, i.e. that a concentration gradient in one of the coupled pair of shear flows induces a flux in the other flow. The way that this arises is that the residual concentration variation across the flow is non-local (i.e. extends into both parts):

$$c' = -f^{(+)} \tfrac{1}{2}(1 - \xi) \partial_x c^{(+)} - f^{(-)} \tfrac{1}{2}(1 + \xi) \partial_x c^{(-)}, \tag{A 6}$$

where

$$f^{(+)} = f_0 + f_1 \, \text{sgn} \, (u_{01}) \left[\frac{1+\xi}{1-\xi} \right]^{\frac{1}{2}}, \tag{A 7a}$$

$$f^{(-)} = f_0 + f_1 \, \text{sgn} \, (u_{01}) \left[\frac{1-\xi}{1+\xi} \right]^{\frac{1}{2}}. \tag{A 7b}$$

The associated flux involves a weighted average:

$$\overline{u\psi^{(+)}c'} = - D^{++} \partial_x c^{(+)} - D^{+-} \partial_x c^{(-)}, \tag{A 8a}$$

$$\overline{u\psi^{(-)}c'} = - D^{-+} \partial_x c^{(+)} - D^{--} \partial_x c^{(-)}, \tag{A 8b}$$

where $\psi^{(+)}, \psi^{(-)}$ are defined as in (A 7a, b) and the diffusivities are given by (A 5a–d).

It happens that for parallel-plate heat exchangers, as studied in §6, the asymmetry implies that the diagonal diffusivities are equal:

$$D^{++} = D^{--} = \frac{\overline{u}^2 d^2}{\kappa} \left[\frac{13}{70} - \frac{3}{5\pi^2} - \frac{18}{\pi^4} + \frac{72}{\pi^6} \right] = 0.01503 \frac{\overline{u}^2 d^2}{\kappa}, \tag{A 9a}$$

and the off-diagonal diffusivities are zero:

$$D^{+-} = D^{-+} = 0. \tag{A 9b}$$

Hence in this case the two approaches are equivalent. Indeed, figures 4 (a, b) could have been computed from the explicit solution given by Chikwendu & Ojiakor (1985, §9) instead of from the NAG computer program DO3 PGF.

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